

Approximate Hermitian-Yang-Mills structures and semistability for Higgs bundles. I: Generalities and the one-dimensional case.

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Abstract

We review the notions of (weak) Hermitian-Yang-Mills structure and approximate Hermitian-Yang-Mills structure for Higgs bundles. Then, we construct the Donaldson functional for Higgs bundles over compact Kähler manifolds and we present some basic properties of it. In particular, we show that its gradient flow can be written in terms of the mean curvature of the Hitchin-Simpson connection. We also study some properties of the solutions of the evolution equation associated with that functional. Next, we study the problem of the existence of approximate Hermitian-Yang-Mills structures and its relation with the algebro-geometric notion of semistability and we show that for a compact Riemann surface, the notion of approximate Hermitian-Yang-Mills structure is in fact the differential-geometric counterpart of the notion of semistability. Finally, we review the notion of admissible Hermitian structure on a torsion-free Higgs sheaf and define the Donaldson functional for such an object.

1 Introduction

In complex geometry, the Hitchin-Kobayashi correspondence asserts that the notion of (Mumford-Takemoto) stability, originally introduced in algebraic geometry, has a differential-geometric equivalent in terms of special metrics. In its classical version, this correspondence is established for holomorphic vector bundles over compact Kähler manifolds and says that such bundles are polystable if and only if they admit an Hermitian-Einstein¹ structure. This correspondence is also true for Higgs bundles.

The history of this correspondence starts in 1965, when Narasimhan and Seshadri [12] proved that a holomorphic bundle on a Riemann surface is stable if and only if it corresponds to a projective irreducible representation of the fundamental group of the surface. Then, in the 80's Kobayashi [8] introduced

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¹In the literature Hermitian-Einstein, Einstein-Hermite and Hermitian-Yang-Mills are all synonymous. Sometimes, also the terminology Hermitian-Yang-Mills-Higgs is used [5].

for the first time the notion of Hermitian-Einstein structure in a holomorphic vector bundle, as a generalization of a Kähler-Einstein metric in the tangent bundle. Shortly after, Kobayashi [9] and Lübke [15] proved that a bundle with an irreducible Hermitian-Einstein structure must be necessarily stable. Donaldson in [16] showed that the result of Narasimhan and Seshadri [12] can be formulated in terms of metrics and showed that the concepts of polystability and Hermitian-Einstein metrics are equivalent for holomorphic vector bundles over a compact Riemann surface. Then, Kobayashi and Hitchin conjectured that the equivalence should be true in general for holomorphic vector bundles over Kähler manifolds. However, the route starting from stability and showing the existence of special structures in higher dimensions took some time.

The existence of Hermitian-Einstein structures in a stable holomorphic vector bundle was proved by Donaldson for projective algebraic surfaces in [17] and for projective algebraic manifolds in [18]. Finally, Uhlenbeck and Yau showed this for general compact Kähler manifolds in [11] using some techniques from analysis and Yang-Mills theory. Hitchin [1], while studying the self-duality equations over a compact Riemann surface, introduced the notion of Higgs field, and showed that the result of Donaldson for Riemann surfaces could be modified to include the presence of a Higgs field. Following the results of Hitchin, Simpson in [2] defined a Higgs bundle to be a holomorphic bundle together with a Higgs field and proved the Hitchin-Kobayashi correspondence for such an object. As an application of this correspondence, Simpson [2], [3] studied in detail a one-to-one correspondence between stable Higgs bundles over compact Kähler manifolds with vanishing Chern classes and representations of the fundamental group of the Kähler manifold.

The Hitchin-Kobayashi correspondence has been further extended in several directions. Simpson [2] studied the Higgs case for non-compact Kähler manifolds satisfying some additional requirements and Lübke and Teleman [14] studied the correspondence for compact complex manifolds. Bando and Siu [19] extended this correspondence for torsion-free sheaves over compact Kähler manifolds and introduced the notion of admissible Hermitian metric for such objects. Following the ideas of Bando and Siu, Biswas and Schumacher [5] generalized this to the Higgs case.

In [17] and [18], Donaldson introduced for the first time a functional (which is known as the Donaldson functional) and later Simpson [2] defined this functional for Higgs bundles. Kobayashi in [10] constructed the same functional in a different form and showed that it played a fundamental role in a possible extension of the Hitchin-Kobayashi correspondence. In fact, using that functional he proved that for holomorphic vector bundles over projective algebraic manifolds, the counterpart of semistability is the notion of approximate Hermitian-Einstein structure.

In this article we show that for a Higgs bundle on a compact Riemann surface, there is a correspondence between semistability and the existence of approximate Hermitian-Yang-Mills structures. We do this by using a Donaldson functional approach analogous to that of Kobayashi [10]. This result covers the classical case if we take the Higgs field equal to zero.

The correspondence between semistability and the existence of approximate Hermitian-Einstein metrics in the ordinary case has been studied recently in [22] using a technique developed by Buchdahl [20] for the desingularization of sheaves in the case of compact complex surfaces. In a future work, following [5], we will study in detail the notion of admissible Hermitian metric on a Higgs sheaf, the Donaldson functional for torsion-free sheaves and the correspondence between semistability and the existence of approximate Hermitian-Yang-Mills metrics for Higgs bundles on compact Kähler manifolds of higher dimensions.

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2 Preliminaries

We start with some basic definitions. Let X be an n -dimensional compact Kähler manifold with ω its Kähler form and let Ω_X^1 be the cotangent sheaf to X , i.e., it is the sheaf of holomorphic one-forms on X . A Higgs sheaf \mathfrak{E} over X is a coherent sheaf E over X , together with a morphism $\phi : E \rightarrow E \otimes \Omega_X^1$ of \mathcal{O}_X -modules, such that the morphism $\phi \wedge \phi : E \rightarrow E \otimes \Omega_X^2$ vanishes. The morphism ϕ is called the Higgs field of \mathfrak{E} . A Higgs sheaf \mathfrak{E} is said to be torsion-free (resp. reflexive, locally free, normal, torsion) if the sheaf E is torsion-free (resp. reflexive, locally free, normal, torsion). A Higgs subsheaf \mathfrak{F} of \mathfrak{E} is a subsheaf F of E such that $\phi(F) \subset F \otimes \Omega_X^1$. A Higgs bundle \mathfrak{E} is just a Higgs sheaf in which the sheaf E is a locally free \mathcal{O}_X -module.

Let \mathfrak{E}_1 and \mathfrak{E}_2 be two Higgs sheaves over a compact Kähler manifold X . A morphism between \mathfrak{E}_1 and \mathfrak{E}_2 is a map $f : E_1 \rightarrow E_2$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi_1} & E_1 \otimes \Omega_X^1 \\ \downarrow f & & \downarrow f \otimes \text{Id} \\ E_2 & \xrightarrow{\phi_2} & E_2 \otimes \Omega_X^1 \end{array}$$

commutes. We will denote such a morphism by $f : \mathfrak{E}_1 \rightarrow \mathfrak{E}_2$. A sequence of Higgs sheaves is a sequence of their corresponding coherent sheaves where each map is a morphism of Higgs sheaves. A short exact sequence of Higgs sheaves (also called an extension of Higgs sheaves or a Higgs extension [3], [6]) is defined in the obvious way.

We define the degree $\deg \mathfrak{E}$ and rank $\text{rk } \mathfrak{E}$ of a Higgs sheaf simply as the degree and rank of the sheaf E . If the rank is positive, we introduce the quotient

$\mu(\mathfrak{E}) = \deg \mathfrak{E} / \text{rk } \mathfrak{E}$ and call it the slope of the Higgs sheaf \mathfrak{E} . In a similar way as in the ordinary case (see for instance [10], [11], [13], [19]) there is a notion of stability for Higgs sheaves, which depends on the Kähler form and makes reference only to Higgs subsheaves [2], [3], [4], [5], [6]. Namely, a Higgs sheaf \mathfrak{E} is said to be ω -stable (resp. ω -semistable) if it is torsion-free and for any Higgs subsheaf \mathfrak{F} with $0 < \text{rk } \mathfrak{F} < \text{rk } \mathfrak{E}$ and torsion-free quotient, one has the inequality $\mu(\mathfrak{F}) < \mu(\mathfrak{E})$ (resp. \leq). We say that a Higgs sheaf is ω -polystable if it decomposes into a direct sum of ω -stable Higgs sheaves all having the same slope.

Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle of rank r over X and let ω be the Kähler form of X . Using the Chern connection $D_{(E,h)}$ of E and the Higgs field ϕ one defines the Hitchin-Simpson connection on \mathfrak{E} by:

$$\mathcal{D}_{(\mathfrak{E},h)} = D_{(E,h)} + \phi + \bar{\phi}_h, \quad (1)$$

where $\bar{\phi}_h$ is the adjoint of the Higgs field with respect to the Hermitian structure h , that is, it is defined by the formula $h(\bar{\phi}_h s, s') = h(s, \phi s')$ with s, s' sections of the Higgs bundle. The curvature of the Hitchin-Simpson connection is then given by $\mathcal{R}_{(\mathfrak{E},h)} = \mathcal{D}_{(\mathfrak{E},h)} \circ \mathcal{D}_{(\mathfrak{E},h)}$ and hence

$$\mathcal{R}_{(\mathfrak{E},h)} = R_{(E,h)} + D'_{(E,h)}(\phi) + D''_{(E,h)}(\bar{\phi}_h) + [\phi, \bar{\phi}_h]. \quad (2)$$

We say that the pair (\mathfrak{E}, h) is Hermitian flat, if the curvature $\mathcal{R}_{(\mathfrak{E},h)}$ vanishes. We denote by $\text{Herm}(\mathfrak{E})$ the space of Hermitian forms in \mathfrak{E} and by $\text{Herm}^+(\mathfrak{E})$ the space of Hermitian structures (i.e., positive definite Hermitian forms) in \mathfrak{E} . For any Hermitian structure h it is possible to identify $\text{Herm}(\mathfrak{E})$ with the tangent space of $\text{Herm}^+(\mathfrak{E})$ at that h (see [10] for details). That is

$$\text{Herm}(\mathfrak{E}) = T_h \text{Herm}^+(\mathfrak{E}). \quad (3)$$

If v denotes an element in $\text{Herm}(\mathfrak{E})$, one defines the endomorphism $h^{-1}v$ of \mathfrak{E} by the formula

$$v(s, s') = h(s, h^{-1}v s'), \quad (4)$$

where s, s' are sections of \mathfrak{E} . We define a Riemannian structure in $\text{Herm}^+(\mathfrak{E})$ via this identification. Namely, for any v, v' in $\text{Herm}(\mathfrak{E})$ we define

$$(v, v')_h = \int_X \text{tr}(h^{-1}v \cdot h^{-1}v') \omega^n / n!. \quad (5)$$

The Higgs field ϕ can be considered as a section of $\text{End } E \otimes \Omega_X^1$ and hence we have a natural dual morphism $\phi^* : E^* \rightarrow E^* \otimes \Omega_X^1$. From this it follows that $\mathfrak{E}^* = (E^*, \phi^*)$ is a Higgs bundle. On the other hand, if Y is another compact Kähler manifold and $f : Y \rightarrow X$ is a holomorphic map, the pair defined by $f^*\mathfrak{E} = (f^*E, f^*\phi)$ is also a Higgs bundle. We have also some natural properties associated to tensor products and direct sums. In particular we have

Proposition 2.1 *Let \mathfrak{E}_1 and \mathfrak{E}_2 two Higgs bundles with Higgs fields ϕ_1 and ϕ_2 respectively. Then*

- (i) *The pair $\mathfrak{E}_1 \otimes \mathfrak{E}_2 = (E_1 \otimes E_2, \phi)$ is a Higgs bundle with $\phi = \phi_1 \otimes I_2 + I_1 \otimes \phi_2$.*
- (ii) *If $\text{pr}_i : E_1 \oplus E_2 \rightarrow E_i$ with $i = 1, 2$ denote the natural projections, then $\mathfrak{E}_1 \oplus \mathfrak{E}_2 = (E_1 \oplus E_2, \phi)$ is a Higgs bundle with $\phi = \text{pr}_1^* \phi_1 + \text{pr}_2^* \phi_2$.*

In a similar form as in the ordinary case [10], [13], [11], we have a notion of Hermitian-Einstein structure for Higgs bundles [2], [3]. Let us consider the usual star operator $*$: $A^{p,q} \rightarrow A^{n-q,n-p}$ and the operator $L : A^{p,q} \rightarrow A^{p+1,q+1}$ defined by $L\varphi = \omega \wedge \varphi$, where φ is a form on X of type (p, q) . Then we define, as usual, $\Lambda = *^{-1} \circ L \circ *$: $A^{p,q} \rightarrow A^{p-1,q-1}$. Consider now a metric h in $\text{Herm}^+(\mathfrak{E})$ (i.e., h is an Hermitian structure on \mathfrak{E}), associated with this metric we have a Hitchin-Simpson curvature $\mathcal{R}_{(\mathfrak{E},h)}$. We can define the mean curvature of the Hitchin-Simpson connection, just by contraction of this curvature with the operator $i\Lambda$. In other words,

$$\mathcal{K}_{(\mathfrak{E},h)} = i\Lambda\mathcal{R}_{(\mathfrak{E},h)}. \quad (6)$$

The mean curvature is an endomorphism² in $\text{End}(E)$. We say that h is a *weak Hermitian-Yang-Mills structure* with factor γ for \mathfrak{E} if

$$\mathcal{K}_{(\mathfrak{E},h)} = \gamma \cdot I \quad (7)$$

where γ is a real function defined on X and I is the identity endomorphism on E . From this definition it follows that if h is a weak Hermitian-Yang-Mills structure with factor γ for \mathfrak{E} , then the dual metric h^* is a weak Hermitian-Yang-Mills structure for the dual bundle \mathfrak{E}^* and also, that any metric h on a Higgs line bundle is necessarily a weak Hermitian-Yang-Mills structure. As in the ordinary case, also for Higgs bundles we have some simple properties related with the notion of weak Hermitian-Yang-Mills structure, in particular, from the usual formulas for the curvature of tensor products and direct sums we have the following

Proposition 2.2 (i) *If h_1 and h_2 are two weak Hermitian-Yang-Mills structures with factors γ_1 and γ_2 for Higgs bundles \mathfrak{E}_1 and \mathfrak{E}_2 , respectively, then $h_1 \otimes h_2$ is a weak Hermitian-Yang-Mills structure with factor $\gamma_1 + \gamma_2$ for the tensor product bundle $\mathfrak{E}_1 \otimes \mathfrak{E}_2$.*

(ii) *The metric $h_1 \oplus h_2$ is a weak Hermitian-Yang-Mills structure with factor γ for the Whitney sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ if and only if both metrics h_1 and h_2 are weak Hermitian-Yang-Mills structures with the same factor γ for \mathfrak{E}_1 and \mathfrak{E}_2 , respectively.*

If we have a weak Hermitian-Yang-Mills structure in which the factor $\gamma = c$ is constant, we say that h is an *Hermitian-Yang-Mills structure* with factor c for \mathfrak{E} . From Proposition 2.2 and this definition we get

Corollary 2.3 *Let $h \in \text{Herm}^+(\mathfrak{E})$ be a (weak) Hermitian-Yang-Mills structure with factor γ for the Higgs bundle \mathfrak{E} . Then*

(i) *The induced Hermitian metric on the tensor product $\mathfrak{E}^{\otimes p} \otimes \mathfrak{E}^{*\otimes q}$ is a (weak) Hermitian-Yang-Mills structure with factor $(p - q)\gamma$.*

(ii) *The induced Hermitian metric on $\bigwedge^p \mathfrak{E}$ is a (weak) Hermitian-Yang-Mills structure with factor $p\gamma$ for every $p \leq r = \text{rk}\mathfrak{E}$.*

In general, if h is a weak Hermitian-Yang-Mills structure with factor γ , the slope of \mathfrak{E} can be written in terms of γ . To be precise, we obtain

²If we consider a local frame field $\{e_i\}_{i=1}^r$ for \mathfrak{E} and a local coordinate system $\{z_\alpha\}_{\alpha=1}^n$ of X , the components of the mean curvature are given by $\mathcal{K}_j^i = \omega^{\alpha\bar{\beta}} \mathcal{R}_{j\alpha\bar{\beta}}^i$.

Proposition 2.4 *If $h \in \text{Herm}^+(\mathfrak{E})$ is a weak Hermitian-Yang-Mills structure with factor γ , then*

$$\mu(\mathfrak{E}) = \frac{1}{2n\pi} \int_X \gamma \omega^n. \quad (8)$$

Proof: Let \mathcal{R} be the Hitchin-Simpson curvature of \mathfrak{E} , then in general we have the identity

$$in\mathcal{R} \wedge \omega^{n-1} = \mathcal{K} \omega^n. \quad (9)$$

Now, by hypothesis h is a weak Hermitian-Yang-Mills structure with factor γ , then taking the trace of (9) and integrating over X we obtain³

$$\deg \mathfrak{E} = \frac{r}{2n\pi} \int_X \gamma \omega^n, \quad (10)$$

where r is the rank of \mathfrak{E} . Q.E.D.

Consider now a real positive function $a = a(x)$ on X , then $h' = ah$ defines another Hermitian metric on \mathfrak{E} . Since h' is a conformal change of h , we have in particular $\bar{\phi}_{h'} = \bar{\phi}_h$. Then, from (9) we obtain

$$\mathcal{K}' \omega^n = in(R' + [\phi, \bar{\phi}_{h'}]) \wedge \omega^{n-1} = (K' + i\Lambda[\phi, \bar{\phi}_h]) \omega^n. \quad (11)$$

Now, defining $\square_0 = i\Lambda d''d'$ (see [10] for details) we have $K' = K + \square_0(\log a)$ and hence using the identity (11) we get

$$\mathcal{K}' \omega^n = \mathcal{K} \omega^n + \square_0(\log a) \omega^n. \quad (12)$$

From this we conclude the following

Lemma 2.5 *Let h be a weak Hermitian-Yang-Mills structure with factor γ for \mathfrak{E} and let a be a real positive definite function on X , then $h' = ah$ is a weak Hermitian-Yang-Mills structure with factor $\gamma' = \gamma + \square_0(\log a)$.*

Making use of Lemma 2.5 we can define a constant c which plays an important role in the definition of the Donaldson functional. Such a constant c is an average of the factor γ of a weak Hermitian-Yang-Mills structure. Namely

Proposition 2.6 *If $h \in \text{Herm}^+(\mathfrak{E})$ is a weak Hermitian-Yang-Mills structure with factor γ , then there exists a conformal change $h' = ah$ such that h' is an Hermitian-Yang-Mills structure with constant factor c , given by*

$$c \int_X \omega^n = \int_X \gamma \omega^n. \quad (13)$$

Such a conformal change is unique up to homothety.

Proof: Let c be as in (13), then

$$\int_X (c - \gamma) \omega^n = 0. \quad (14)$$

³We consider here the integral $\frac{i}{2\pi} \int_X \text{tr } \mathcal{R} \wedge \omega^{n-1}$. Notice that only the $(1,1)$ part of the curvature \mathcal{R} makes a real contribution in such an integral and since $\text{tr } [\phi, \bar{\phi}]$ is identically zero, that integral must be the degree of the holomorphic bundle E and hence it is equal to $\deg \mathfrak{E}$.

It is sufficient to prove that there is a function u satisfying the equation

$$\square_0 u = c - \gamma, \quad (15)$$

where, as we said before $\square_0 = i\Lambda d''d'$. Because if this holds, then by applying Lemma 2.5 with the function $a = e^u$ the result follows.

Now, from Hodge theory we know that the equation (15) has a solution if and only if the function $c - \gamma$ is orthogonal to all \square_0 -harmonic functions. Since X is compact, a function is \square_0 -harmonic if and only if it is constant. But (14) says that $c - \gamma$ is orthogonal to the constant functions and hence the equation (15) has a solution u . Finally, the uniqueness follows from the fact that \square_0 -harmonic functions are constant. Q.E.D.

Since every weak Hermitian-Yang-Mills structure can be transformed into an Hermitian-Yang-Mills structure using a conformal change of the metric, without loss of generality we avoid using weak structures and work directly with Hermitian-Yang-Mills structures.

3 Approximate Hermitian-Yang-Mills structures

As we have seen in the preceding section, if we have an Hermitian-Yang-Mills structure with factor c , this constant can be evaluated directly from (8) and we have

$$c = \frac{2\pi \mu(\mathfrak{E})}{(n-1)! \operatorname{vol} X}. \quad (16)$$

On the other hand, regardless if we have an Hermitian-Yang-Mills structure or not on \mathfrak{E} , we can always define a constant c just by (16). Introduced in such a way, c depends only on $c_1(\mathfrak{E})$ and the cohomology class of ω and not on the metric h . We define the length of the endomorphism $\mathcal{K} - cI$ by the formula

$$|\mathcal{K} - cI|^2 = \operatorname{tr}[(\mathcal{K} - cI) \circ (\mathcal{K} - cI)]. \quad (17)$$

We say that a Higgs bundle \mathfrak{E} over a compact Kähler manifold X admits an *approximate Hermitian-Yang-Mills structure* if for any $\epsilon > 0$ there exists a metric h (which depends on ϵ) such that

$$\max |\mathcal{K} - cI| < \epsilon. \quad (18)$$

From the above definition it follows that \mathfrak{E}^* admits an approximate Hermitian-Yang-Mills structure if \mathfrak{E} does. This notion satisfies some simple properties with respect to tensor products and direct sums.

Proposition 3.1 *If \mathfrak{E}_1 and \mathfrak{E}_2 admit approximate Hermitian-Yang-Mills structures, so does their tensor product $\mathfrak{E}_1 \otimes \mathfrak{E}_2$. Furthermore if $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, so does their Whitney sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2$.*

Proof: Assume that \mathfrak{E}_1 and \mathfrak{E}_2 admit approximate Hermitian-Yang-Mills structures with factors c_1 and c_2 respectively and let $\epsilon > 0$. Then, there exist h_1 and h_2 such that

$$\max_X |\mathcal{K}_1 - c_1 I_1| < \frac{\epsilon}{2\sqrt{r_2}}, \quad \max_X |\mathcal{K}_2 - c_2 I_2| < \frac{\epsilon}{2\sqrt{r_1}},$$

where r_1, r_2 and I_1, I_2 are the ranks and the identity endomorphisms of \mathfrak{E}_1 and \mathfrak{E}_2 respectively. Now, let \mathcal{K} be the Hitchin-Simpson mean curvature of $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ associated with the metric $h = h_1 \otimes h_2$. Then, by defining $c = c_1 + c_2$ and $I = I_1 \otimes I_2$ it follows

$$\begin{aligned} |\mathcal{K} - cI| &= |\mathcal{K}_1 \otimes I_2 + I_1 \otimes \mathcal{K}_2 - (c_1 + c_2)I_1 \otimes I_2| \\ &\leq |(\mathcal{K}_1 - c_1 I_1) \otimes I_2| + |I_1 \otimes (\mathcal{K}_2 - c_2 I_2)| \\ &\leq \sqrt{r_2} |\mathcal{K}_1 - c_1 I_1| + \sqrt{r_1} |\mathcal{K}_2 - c_2 I_2| \\ &< \epsilon \end{aligned}$$

and hence the tensor product $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ admits an approximate Hermitian-Yang-Mills structure.

On the other hand, if $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, necessarily the constants c_1 and c_2 coincide. Then, taking this time $c = c_1 = c_2$, $I = I_1 \oplus I_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, we have

$$\begin{aligned} |\mathcal{K} - cI| &= |\mathcal{K}_1 \oplus \mathcal{K}_2 - c(I_1 \oplus I_2)| \\ &= \sqrt{\text{tr}(\mathcal{K}_1 - c_1 I_1)^2 + \text{tr}(\mathcal{K}_2 - c_2 I_2)^2} \\ &\leq |\mathcal{K}_1 - c_1 I_1| + |\mathcal{K}_2 - c_2 I_2|. \end{aligned}$$

From this inequality it follows that $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ admits an approximate Hermitian-Yang-Mills structure. Q.E.D.

Corollary 3.2 *If \mathfrak{E} admits an approximate Hermitian-Yang-Mills structure, so do the tensor product bundle $\mathfrak{E}^{\otimes p} \otimes \mathfrak{E}^{*\otimes q}$ and the exterior product bundle $\bigwedge^p \mathfrak{E}$ whenever $p \leq r$.*

Finally, in a similar way as in the classical case, we have a version of the Bogomolov-Lübke inequality also for Higgs bundles admitting an approximate Hermitian-Yang-Mills structure (see [10], [21] for details). To be precise, we obtain

Theorem 3.3 *Let \mathfrak{E} be a Higgs bundle over a compact Kähler manifold X and suppose that \mathfrak{E} admits an approximate Hermitian-Yang-Mills structure, then*

$$\int_X [2r c_2(\mathfrak{E}) - (r-1)c_1(\mathfrak{E})^2] \wedge \omega^{n-2} \geq 0. \quad (19)$$

Proof: Assume that \mathfrak{E} admits an approximate Hermitian-Yang-Mills structure. Let $\epsilon > 0$ and suppose h_ϵ is a metric on \mathfrak{E} satisfying (18). Then, we have closed $2k$ -forms $c_k(\mathfrak{E}, h_\epsilon)$ representing the k -th Chern classes. From [10], Ch.IV, we obtain

$$(2r c_2(\mathfrak{E}, h_\epsilon) - (r-1) c_1(\mathfrak{E}, h_\epsilon)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} = [r(|R_\epsilon|^2 - |K_\epsilon|^2) + \sigma_\epsilon^2 - |\rho_\epsilon|^2] \frac{\omega^n}{n!}$$

where the quantities on the right-hand side are associated to the metric h_ϵ and are given by $|K_\epsilon|^2 = \text{tr } K_\epsilon^2$ and $\sigma_\epsilon = \text{tr } K_\epsilon$ and

$$|R_\epsilon|^2 = \sum_{i,j,\alpha,\beta} |(R_\epsilon)_{j\alpha\bar{\beta}}^i|^2, \quad |\rho_\epsilon|^2 = \sum_{i,\alpha,\beta} |(R_\epsilon)_{i\alpha\bar{\beta}}^i|^2.$$

Now, one has $r|R_\epsilon|^2 \geq |\rho_\epsilon|^2$, and hence integrating over X we obtain

$$\int_X (2r c_2(\mathfrak{E}, h_\epsilon) - (r-1) c_1(\mathfrak{E}, h_\epsilon)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq \int_X [\sigma_\epsilon^2 - r |K_\epsilon|^2] \frac{\omega^n}{n!}. \quad (20)$$

Since h_ϵ is an approximate Hermitian-Yang-Mills structure, we have

$$\epsilon^2 > |K_\epsilon - cI|^2 = |K_\epsilon|^2 - 2c\sigma_\epsilon + c^2r. \quad (21)$$

On the other hand,

$$\begin{aligned} |K_\epsilon|^2 &= \text{tr} [(K_\epsilon + i\Lambda[\phi, \bar{\phi}_\epsilon]) \cdot (K_\epsilon + i\Lambda[\phi, \bar{\phi}_\epsilon])] \\ &= |K_\epsilon|^2 + 2i\Lambda \text{tr} [K_\epsilon \cdot [\phi, \bar{\phi}_\epsilon]] + (i\Lambda)^2 \text{tr} [[\phi, \bar{\phi}_\epsilon]^2] \\ &= |K_\epsilon|^2 + 2i\Lambda \text{tr} [K_\epsilon \cdot [\phi, \bar{\phi}_\epsilon]]. \end{aligned}$$

Now, $K_\epsilon = cI + \epsilon A$ with A a self-adjoint endomorphism of E and hence we can estimate the term involving the trace in the last expression as

$$\text{tr} [K_\epsilon \cdot [\phi, \bar{\phi}_\epsilon]] = c \text{tr} [\phi, \bar{\phi}_\epsilon] + \epsilon \text{tr} [A \cdot [\phi, \bar{\phi}_\epsilon]] = \epsilon \eta \quad (22)$$

where the $(1,1)$ -form $\eta = \text{tr} [A \cdot [\phi, \bar{\phi}_\epsilon]]$. Consequently

$$|K_\epsilon|^2 = |K_\epsilon|^2 + 2\epsilon (i\Lambda\eta). \quad (23)$$

Finally, from (21) and (23) it follows

$$\sigma_\epsilon^2 - r |K_\epsilon|^2 > (\sigma_\epsilon - cr)^2 + f(\epsilon)$$

where $f(\epsilon) = r\epsilon(2(i\Lambda\eta) - \epsilon)$. Then, by replacing this last expression in (20) we conclude

$$\int_X (2r c_2(\mathfrak{E}, h_\epsilon) - (r-1) c_1(\mathfrak{E}, h_\epsilon)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} > \int_X f(\epsilon) \frac{\omega^n}{n!}. \quad (24)$$

Now, the integral on the left-hand side is independent of the metric h_ϵ . On the other hand, the above inequality holds for all $\epsilon > 0$ and clearly $f(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Therefore, one has the inequality (19) if \mathfrak{E} admits an approximate Hermitian-Yang-Mills metric. Q.E.D.

We want to construct a functional \mathcal{L} on $\text{Herm}^+(\mathfrak{E})$, that will be called Donaldson's functional and whose gradient is related with the mean curvature of the Hitchin-Simpson connection. The construction of this functional is in certain way similar to the ordinary case. However, there are some differences, which in essence are due to the extra terms involving the Higgs field ϕ in the expression for the curvature (2).

4 Donaldson's functional

Given two Hermitian structures h, k in $\text{Herm}^+(\mathfrak{E})$, we connect them by a curve h_t , $0 \leq t \leq 1$, in $\text{Herm}^+(\mathfrak{E})$ so that $k = h_0$ and $h = h_1$. We set

$$Q_1(h, k) = \log(\det(k^{-1}h)), \quad Q_2(h, k) = i \int_0^1 \text{tr}(v_t \cdot \mathcal{R}_t) dt, \quad (25)$$

where $v_t = h_t^{-1} \partial_t h_t$ and \mathcal{R}_t denotes the curvature of the Hitchin-Simpson connection associated with h_t . Notice that $Q_1(h, k)$ does not involve the curve (in fact, it is the same functional of the ordinary case). On the other hand, the definition of $Q_2(h, k)$ uses explicitly the curve and differs from the ordinary case because of the extra terms in (2). We define the Donaldson functional by

$$\mathcal{L}(h, k) = \int_X \left[Q_2(h, k) - \frac{c}{n} Q_1(h, k) \omega \right] \wedge \omega^{n-1} / (n-1)!, \quad (26)$$

where c is the constant given by

$$c = \frac{2\pi \mu(\mathfrak{E})}{(n-1)! \text{vol } X}. \quad (27)$$

Notice that the components of $(2, 0)$ and $(0, 2)$ type of \mathcal{R}_t do not contribute to $\mathcal{L}(h, k)$. This means that, in practice, it is enough to consider in the definition of $Q_2(h, k)$ just components of $(1, 1)$ -type⁴.

The following Lemma and the subsequent Proposition are straightforward generalizations of a result of Kobayashi (see [10], Ch.VI, Lemma 3.6) to the Higgs case. Part of the proof is similar to the proof presented in [10], however some differences arise because of the term involving the commutator in the Hitchin-Simpson curvature.

Lemma 4.1 *Let $h_t, a \leq t \leq b$, be any differentiable curve in $\text{Herm}^+(\mathfrak{E})$ and k any fixed Hermitian structure of \mathfrak{E} . Then, the $(1, 1)$ -component of*

$$i \int_a^b \text{tr}(v_t \cdot \mathcal{R}_t) dt + Q_2(h_a, k) - Q_2(h_b, k) \quad (28)$$

is an element in $d' A^{0,1} + d'' A^{1,0}$.

Proof: Following [10], we consider the domain Δ in \mathbb{R}^2 defined by

$$\Delta = \{(t, s) | a \leq t \leq b, 0 \leq s \leq 1\} \quad (29)$$

and let $h : \Delta \rightarrow \text{Herm}^+(\mathfrak{E})$ be a smooth mapping such that $h(t, 0) = k, h(t, 1) = h_t$ for $a \leq t \leq b$, let $h(a, s)$ and $h(b, s)$ line segments curves⁵ from k to h_a and respectively from k to h_b . We define the endomorphisms $u = h^{-1} \partial_s h$, $v = h^{-1} \partial_t h$ and we put

$$\mathcal{R} = d''(h^{-1} d' h) + [\phi, \bar{\phi}_h] \quad (30)$$

and $\Psi = i \text{tr}[h^{-1} \tilde{d} h \mathcal{R}]$, where $\tilde{d} h = \partial_s h ds + \partial_t h dt$ is considered as the exterior differential of h in the domain Δ . It is convenient to rewrite Ψ in the form

$$\Psi = i \text{tr}[(u ds + v dt) \mathcal{R}]. \quad (31)$$

⁴In other words, in computations involving integration over X , we can always replace the curvature by $\mathcal{R}_t^{1,1} = R_t + [\phi, \bar{\phi}_{h_t}]$.

⁵Notice that we have a simple expression for line segments curves from k to h_t given by $h(t, s) = s h_t + (1-s) k$.

Applying the Stokes formula to Ψ (which is considered here as a 1-form in the domain Δ) we get

$$\int_{\Delta} \tilde{d}\Psi = \int_{\partial\Delta} \Psi. \quad (32)$$

The right hand side of the above expression can be computed straightforward from definition. In fact, after a short computation we obtain

$$\int_{\partial\Delta} \Psi = i \int_a^b \text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) dt + Q_2^{1,1}(h_a, k) - Q_2^{1,1}(h_b, k). \quad (33)$$

Therefore, we need to show that the left hand side of (32) is an element in $d'A^{0,1} + d''A^{1,0}$, and hence, it suffices to show that $\tilde{d}\Psi \in d'A^{0,1} + d''A^{1,0}$.

Now, from the definition of Ψ we have

$$\begin{aligned} \tilde{d}\Psi &= i \text{tr}[\tilde{d}(u ds + v dt) \mathcal{R} - (u ds + v dt) \tilde{d}\mathcal{R}] \\ &= i \text{tr}[(\partial_s v - \partial_t u) \mathcal{R} - u \partial_t \mathcal{R} + v \partial_s \mathcal{R}] ds \wedge dt. \end{aligned}$$

On the other hand, a simple computation shows that

$$\partial_t u = -vu + h^{-1} \partial_t \partial_s h, \quad \partial_s v = -uv + h^{-1} \partial_s \partial_t h, \quad (34)$$

$$\partial_t \mathcal{R} = d'' D' v + [\phi, \partial_t \bar{\phi}_h], \quad \partial_s \mathcal{R} = d'' D' u + [\phi, \partial_s \bar{\phi}_h]. \quad (35)$$

Replacing these expressions in the formula for $\tilde{d}\Psi$ and writing $\mathcal{R} = R + [\phi, \bar{\phi}_h]$ we obtain

$$\begin{aligned} \tilde{d}\Psi &= i \text{tr}[(vu - uv)R - u d'' D' v + v d'' D' u] ds \wedge dt \\ &\quad + i \text{tr}[v[\phi, \partial_s \bar{\phi}_h] - u[\phi, \partial_t \bar{\phi}_h] + (vu - uv)[\phi, \bar{\phi}_h]] ds \wedge dt. \end{aligned}$$

The first trace in the expression above does not depend on Higgs field ϕ (in fact, it is the same expression that is found in [10] for the ordinary case). The second trace is identically zero. In order to prove this, we need first to find explicit expressions for $\partial_t \bar{\phi}_h$ and $\partial_s \bar{\phi}_h$. Now (omitting the parameter t for simplicity) we know from [2] that

$$\bar{\phi}_{h_s + \delta s} = u_0^{-1} \bar{\phi}_{h_s} u_0 = \bar{\phi}_{h_s} + u_0^{-1} [\bar{\phi}_{h_s}, u_0] \quad (36)$$

where u_0 is a selfadjoint endomorphism such that $h_{s+\delta s} = h_s u_0$. Now

$$h_{s+\delta s} = h_s + \partial_s h_s \cdot \delta s + \mathcal{O}(\delta s^2) \quad (37)$$

and hence, at first order in δs , we obtain $u_0 = 1 + u \cdot \delta s$ and consequently $\partial_s \bar{\phi}_h = [\bar{\phi}_h, u]$. In a similar way we obtain the formula $\partial_t \bar{\phi}_h = [\bar{\phi}_h, v]$. Therefore, using these relations, the Jacobi identity and the cyclic property of the trace, we see that the second trace is identically zero. On the other hand, the term involving the curvature R can be rewritten in terms of u, v and their covariant derivatives. So, finally we get

$$\tilde{d}\Psi = -i \text{tr}[v D' d'' u + u d'' D' v] ds \wedge dt. \quad (38)$$

As it is shown in [10], defining the (0,1)-form $\alpha = i \text{tr}[v d'' u]$ we obtain

$$\tilde{d}\Psi = -[d' \alpha + d'' \bar{\alpha} + i d'' d' \text{tr}(vu)] ds \wedge dt \quad (39)$$

and hence $\tilde{d}\Psi$ is an element in $d'A^{0,1} + d''A^{1,0}$. Q.E.D.

As a consequence of Lemma 4.1 we have an important result for piecewise differentiable closed curves. Namely, we have

Proposition 4.2 *Let h_t , $\alpha \leq t \leq \beta$, be a piecewise differentiable closed curve in $\text{Herm}^+(\mathfrak{E})$. Then*

$$i \int_{\alpha}^{\beta} \text{tr} \left(v_t \cdot \mathcal{R}_t^{1,1} \right) dt = 0 \mod d'A^{0,1} + d''A^{1,0}. \quad (40)$$

Proof: Let $\alpha = a_0 < a_1 < \dots < a_p = \beta$ be the values of t where h_t is not differentiable. Now take a fixed point k in $\text{Herm}^+(\mathfrak{E})$. Then, Lemma 4.1 applies for each triple $k, h_{a_j}, h_{a_{j+1}}$ with $j = 0, 1, \dots, p-1$ and the result follows. Q.E.D.

Corollary 4.3 *The Donaldson functional $\mathcal{L}(h, h')$ does not depend on the curve joining h and h' .*

Proof: Clearly, from the definition of Q_1

$$Q_1(h, h') + Q_1(h', h) = 0. \quad (41)$$

If γ_1 and γ_2 are two differentiable curves from h to h' and we apply Proposition 4.2 to $\gamma_1 - \gamma_2$, we obtain

$$Q_2^{1,1}(h, h') + Q_2^{1,1}(h', h) = 0 \mod d'A^{0,1} + d''A^{1,0}, \quad (42)$$

and the result follows by integrating over X . Q.E.D.

Proposition 4.4 *For any h in $\text{Herm}^+(\mathfrak{E})$ and any constant $a > 0$, the Donaldson functional satisfies $\mathcal{L}(h, ah) = 0$.*

Proof: Clearly

$$Q_1(h, ah) = \log \det[(ah)^{-1}h] = -r \log a.$$

Now, let $b = \log a$ and consider the curve $h_t = e^{b(1-t)}h$ from ah to h . For this curve $v_t = -bI$ and we have

$$\mathcal{R}_t^{1,1} = d''(h_t^{-1}d'h_t) + [\phi, \bar{\phi}_t] = d''(h^{-1}d'h) + [\phi, \bar{\phi}_t],$$

where $\bar{\phi}_t$ is an abbreviation for $\bar{\phi}_{h_t}$. Therefore, the $(1,1)$ -component of $Q_2(h, ah)$ becomes

$$Q_2^{1,1}(h, ah) = i \int_0^1 \text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) dt = i \int_0^1 \text{tr}[-b(R + [\phi, \bar{\phi}_t])] dt = -ib \text{tr} R$$

and hence, from the above we obtain

$$\frac{c}{n} \int_X Q_1(h, ah) \omega \wedge \omega^{n-1} / (n-1)! = -c b \text{vol} X,$$

$$\begin{aligned} \int_X Q_2(h, ah) \wedge \omega^{n-1} / (n-1)! &= \frac{-ib}{(n-1)!} \int_X \text{tr} R \wedge \omega^{n-1} \\ &= \frac{-2\pi b}{(n-1)!} \deg \mathfrak{E} \end{aligned}$$

and the result follows from the definition of the constant c . Q.E.D.

Lemma 4.5 *For any differentiable curve h_t and any fixed point k in $\text{Herm}^+(\mathfrak{E})$ we have*

$$\partial_t Q_1(h_t, k) = \text{tr}(v_t), \quad (43)$$

$$\partial_t Q_2^{1,1}(h_t, k) = i \text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) \mod d' A^{0,1} + d'' A^{1,0}. \quad (44)$$

Proof: Since k does not depend on t , we get

$$\partial_t Q_1(h_t, k) = \partial_t \log(\det k^{-1}) + \partial_t \log(\det h_t) = \partial_t \log(\det h_t) = \text{tr}(v_t).$$

Considering b in (28) as a variable, and differentiating that expression with respect to b , we obtain the formula. Q.E.D.

By using the above Lemma, we have a formula for the derivative with respect to t of Donaldson's functional

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(h_t, k) &= \int_X \left[i \text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) - \frac{c}{n} \text{tr}(v_t) \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!} \\ &= \int_X [\text{tr}(v_t \cdot \mathcal{K}_t) - c \text{tr}(v_t)] \frac{\omega^n}{n!} \\ &= \int_X \text{tr}[(\mathcal{K}_t - cI)v_t] \frac{\omega^n}{n!}. \end{aligned}$$

Since $v_t = h_t^{-1} \partial_t h_t$ and we can consider the endomorphism \mathcal{K}_t as an Hermitian form by defining $\mathcal{K}_t(s, s') = h_t(s, \mathcal{K}_t s')$, for any fixed Hermitian metric k and any differentiable curve h_t in $\text{Herm}^+(\mathfrak{E})$ we obtain⁶

$$\frac{d}{dt} \mathcal{L}(h_t, k) = (\mathcal{K}_t - c h_t, \partial_t h_t), \quad (45)$$

where \mathcal{K}_t is considered here as a form. For each t , we can consider $\partial_t h_t \in \text{Herm}(\mathfrak{E})$ as a tangent vector of $\text{Herm}^+(\mathfrak{E})$ at h_t . Therefore, the differential $d\mathcal{L}$ of the functional evaluated at $\partial_t h_t$ is given by

$$d\mathcal{L}(\partial_t h_t) = \frac{d}{dt} \mathcal{L}(h_t, k), \quad (46)$$

and hence, the gradient of \mathcal{L} (i.e., the vector field on $\text{Herm}^+(\mathfrak{E})$ dual to the form $d\mathcal{L}$ with respect to the invariant Riemannian metric introduced before) is given by $\nabla \mathcal{L} = \mathcal{K} - c h$. From the above analysis we conclude the following

Theorem 4.6 *Let k be a fixed element in $\text{Herm}^+(\mathfrak{E})$. Then, h is a critical point of \mathcal{L} if and only if $\mathcal{K} - c h = 0$, i.e., if and only if h is an Hermitian-Yang-Mills structure for \mathfrak{E} .*

In order to derive some properties of \mathcal{L} it is convenient to divide the Hitchin-Simpson connection (see [2], [3]) in the form $\mathcal{D}'_h = D'_h + \bar{\phi}_h$ and $\mathcal{D}'' = D'' + \phi$. In fact, using the above decomposition it is not difficult to show that all critical points of \mathcal{L} correspond to an absolute minimum.

⁶Notice that from the definition (4), the endomorphism \mathcal{K}_t can be written formally as $\mathcal{K}_t = h_t^{-1} \mathcal{K}_t(\cdot, \cdot)$ where $\mathcal{K}_t(\cdot, \cdot)$ denotes this time the mean curvature as a form. Therefore, we can express the derivative of the functional as an inner product of the forms $\mathcal{K}_t - c h_t$ and $\partial_t h_t$ as in (5).

Theorem 4.7 *Let k be a fixed Hermitian structure of \mathfrak{E} and \tilde{h} a critical point of $\mathcal{L}(h, k)$, then the Donaldson functional attains an absolute minimum at \tilde{h} .*

Proof: Let $h_t, 0 \leq t \leq 1$, be a differentiable curve such that $h_0 = \tilde{h}$, then we can compute straightforward the second derivative of \mathcal{L}

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{L}(h_t, k) &= \frac{d}{dt} \int_X \operatorname{tr} [(\mathcal{K}_t - cI)v_t] \frac{\omega^n}{n!} \\ &= \int_X \operatorname{tr} [\partial_t \mathcal{K}_t \cdot v_t + (\mathcal{K}_t - cI) \partial_t v_t] \frac{\omega^n}{n!}. \end{aligned}$$

Since h_0 is a critical point of the functional, then $\mathcal{K}_t - cI = 0$ at $t = 0$, and hence

$$\frac{d^2}{dt^2} \mathcal{L}(h_t, k)|_{t=0} = \int_X \operatorname{tr} (\partial_t \mathcal{K}_t \cdot v_t) \frac{\omega^n}{n!} |_{t=0}. \quad (47)$$

On the other hand, $\partial_t \mathcal{K}_t$ can be written in terms of the endomorphism v_t in the following way

$$\begin{aligned} \mathcal{D}'' \mathcal{D}' v_t &= \mathcal{D}'' (D' v_t + [\bar{\phi}_t, v_t]) \\ &= D'' D' v_t + [\phi, D' v_t] + D'' [\bar{\phi}_t, v_t] + [\phi, [\bar{\phi}_t, v_t]], \end{aligned}$$

and since $\partial_t \phi_t = [\bar{\phi}_t, v_t]$ we get

$$\partial_t \mathcal{R}_t^{1,1} = \partial_t R_t + [\phi, \partial_t \bar{\phi}_t] = D'' D' v_t + [\phi, [\bar{\phi}_t, v_t]]. \quad (48)$$

Therefore, taking the trace with respect to ω (i.e., applying the $i\Lambda$ operator) we obtain

$$i\Lambda \mathcal{D}'' \mathcal{D}' v_t = i\Lambda \partial_t \mathcal{R}_t^{1,1} = \partial_t \mathcal{K}_t. \quad (49)$$

Hence, replacing this in the expression for the second derivative of \mathcal{L} we find

$$\frac{d^2}{dt^2} \mathcal{L}(h_t, k)|_{t=0} = \int_X \operatorname{tr} (i\Lambda \mathcal{D}'' \mathcal{D}' v_t \cdot v_t) \frac{\omega^n}{n!} |_{t=0} = \|\mathcal{D}' v_t\|_{t=0}^2, \quad (50)$$

(that is, h_0 must be at least a local minimum of \mathcal{L}). Now suppose in addition that h_1 is an arbitrary element in $\operatorname{Herm}^+(\mathfrak{E})$ and joint them by a geodesic h_t , and hence $\partial_t v_t = 0$. Therefore, for a such a geodesic we have

$$\frac{d^2}{dt^2} \mathcal{L}(h_t, k) = \int_X \operatorname{tr} (\partial_t \mathcal{K}_t \cdot v_t) \frac{\omega^n}{n!}. \quad (51)$$

Following the same procedure we have done before, but this time at t arbitrary, we get for $0 \leq t \leq 1$

$$\frac{d^2}{dt^2} \mathcal{L}(h_t, k) = \|\mathcal{D}' v_t\|_{h_t}^2 \geq 0 \quad (52)$$

(since there is an implicit dependence on t on the right hand side via \mathcal{D}' , we write a subscript h_t in the norm) and it follows that $\mathcal{L}(h_0, k) \leq \mathcal{L}(h_1, k)$. Now if we assume that h_1 is also a critical point of \mathcal{L} , we necessarily obtain the equality. Therefore, it follows that the minimum defined for any critical point of \mathcal{L} is an absolute minimum. Q.E.D.

Let k be a fixed Hermitian structure, then any Hermitian metric h will be of the form ke^v for some section v of $\text{End}(E)$ over X . We can join k to h by the geodesic $h_t = ke^{tv}$ where $0 \leq t \leq 1$ (note that here $v_t = h_t^{-1} \partial_t h_t = v$ is constant, i.e., it does not depend on t). Now, in the proof of Theorem 4.7 we got an expression for the second derivative $\mathcal{L}(h_t, k)$ for any curve h_t . Namely

$$\frac{d^2}{dt^2} \mathcal{L}(h_t, k) = \int_X \text{tr} [\partial_t \mathcal{K}_t \cdot v_t + (\mathcal{K}_t - cI) \partial_t v_t] \frac{\omega^n}{n!}. \quad (53)$$

Notice that in our case, the chosen curve is such that $h_0 = k$, since it is also a geodesic $\partial_t v_t = 0$ we have

$$\frac{d^2}{dt^2} \mathcal{L}(h_t, k) = \int_X \text{tr} (\partial_t \mathcal{K}_t \cdot v) \frac{\omega^n}{n!} = \|\mathcal{D}'v\|_{h_t}^2. \quad (54)$$

Therefore, following [13], the idea is to find a simple expression for $\|\mathcal{D}'v\|_{h_t}^2$ or equivalently for $\|\mathcal{D}''v\|_{h_t}^2$ and to integrate it twice with respect to t . We can do this using local coordinates, in fact, at any point in X we can choose a local frame field so that $h_0 = I$ and $v = \text{diag}(\lambda_1, \dots, \lambda_r)$. In particular, using such a local frame field we have $h_t^{ij} = e^{-\lambda_j t} \delta_{ij}$, and hence (after a short computation) we obtain

$$\|\mathcal{D}''v\|_{h_t}^2 = \int_X \sum_{i,j=1}^r e^{(\lambda_i - \lambda_j)t} |\mathcal{D}''v_j^i|^2 \frac{\omega^{n-1}}{(n-1)!}. \quad (55)$$

Now, at $t = 0$ the functional $\mathcal{L}(h_t, k)$ vanishes and since $k = h_0$ is not necessarily an Hermitian-Yang-Mills structure, we have

$$\frac{d}{dt} \mathcal{L}(h_t, k)|_{t=0} = \int_X \text{tr} [(\mathcal{K}_0 - cI)v] \frac{\omega^n}{n!}. \quad (56)$$

Then, by integrating twice (54) we obtain

$$\mathcal{L}(h_t, k) = t \int_X \text{tr} [(\mathcal{K}_0 - cI)v] \frac{\omega^n}{n!} + \int_X \sum_{i,j=1}^r \psi_t(\lambda_i, \lambda_j) |\mathcal{D}''v_j^i|^2 \frac{\omega^{n-1}}{(n-1)!} \quad (57)$$

where ψ_t is a function given by

$$\psi_t(\lambda_i, \lambda_j) = \frac{e^{(\lambda_i - \lambda_j)t} - (\lambda_i - \lambda_j)t - 1}{(\lambda_i - \lambda_j)^2}. \quad (58)$$

In particular, at $t = 1$ the expression (57) corresponds (up to a constant term) to the definition of Donaldson's functional given by Simpson in [2]. Notice also that if the initial metric $k = h_0$ is Hermitian-Yang-Mills, the first term of the right hand side of (57) vanishes and the functional coincides with the Donaldson functional used by Siu in [13].

5 The evolution equation

For the construction of Hermitian-Yang-Mills structures, the standard procedure is to start with a fixed Hermitian metric h_0 and try to find from it an Hermitian metric satisfying $\mathcal{K} = cI$ using a curve h_t , $0 \leq t < \infty$ (in other words, we try

to find that metric by deforming h_0 through 1-parameter family of Hermitian metrics) and we expect that at $t = \infty$, the metric will be Hermitian-Yang-Mills.

At this point, it is convenient to introduce the operator $\tilde{\square}_h = i\Lambda\mathcal{D}''\mathcal{D}'_h$, which depends on the metric h . Using it, we can rewrite (49) as $\partial_t\mathcal{K}_t = \tilde{\square}_t v_t$, where the subscript t reminds the dependence of the operator on the metric h_t .

As we said before, to get an Hermitian-Yang-Mills metric we want to make $\mathcal{K} - cI$ vanish. Therefore, a natural choice is to go along the global gradient direction of the functional given by the global L^2 -norm of $\mathcal{K}_t - cI$. Therefore, taking the derivative of this functional we obtain

$$\begin{aligned} \frac{d}{dt}\|\mathcal{K}_t - cI\|^2 &= \int_X 2 \operatorname{tr}(\partial_t\mathcal{K}_t \cdot (\mathcal{K}_t - cI)) \frac{\omega^n}{n!} \\ &= 2 \int_X \operatorname{tr}(\tilde{\square}_t v_t \cdot (\mathcal{K}_t - cI)) \frac{\omega^n}{n!} \\ &= 2 \int_X \operatorname{tr}(v_t \cdot \tilde{\square}_t \mathcal{K}_t) \frac{\omega^n}{n!}, \end{aligned}$$

and the equation that naturally emerges (i.e., the associated steepest descent curve) is $v_t = -\tilde{\square}_t \mathcal{K}_t$, or equivalently

$$h_t^{-1} \partial_t h_t = -i\Lambda\mathcal{D}''\mathcal{D}'_h \mathcal{K}_t. \quad (59)$$

Since \mathcal{K}_t is of degree two, the right hand side of the above equation becomes a term of degree four and hence we get at the end a nonlinear equation of degree four. To do the analysis, it is easier to deal with an equation of lower degree. In fact, this is one of the reasons for introducing the Donaldson functional. Following the same argument we did before, but this time using the functional $\mathcal{L}(h_t, k)$ with k fixed, in place of the functional $\|\mathcal{K}_t - cI\|^2$, we end up with a nonlinear equation of degree two (the heat equation), to be more precise, we obtain directly from (45) the equation

$$\partial_t h_t = -(\mathcal{K}_t - ch_t), \quad (60)$$

where this time, \mathcal{K}_t represents the associated two form, and not an endomorphism⁷. Simpson has shown that also for the Higgs case, we have always solutions of the above non linear evolution equation. This was proved in [2] for the non-compact case satisfying some additional conditions. That proof covers the compact Kähler case without any change. Then, from [2] we have the following

Theorem 5.1 *Given an Hermitian structure h_0 on \mathfrak{E} , the non-linear evolution equation*

$$\partial_t h_t = -(\mathcal{K}_t - ch_t) \quad (61)$$

has a unique smooth solution defined for $0 \leq t < \infty$.

In the rest of this section, we study some properties of the solutions of the evolution equation. In particular, we are interested in the study of the mean curvature when the parameter t goes to infinity.

⁷Notice that the equivalent equation involving endomorphisms will be $v_t = -(\mathcal{K}_t - cI)$.

Proposition 5.2 *Let $h_t, 0 \leq t < \infty$, be a 1-parameter family of $\text{Herm}^+(\mathfrak{E})$ satisfying the evolution equation. Then*

(i) *For any fixed Hermitian structure k of \mathfrak{E} , the functional $\mathcal{L}(h_t, k)$ is a monotone decreasing function of t ; that is*

$$\frac{d}{dt}\mathcal{L}(h_t, k) = -\|\mathcal{K}_t - cI\|^2 \leq 0; \quad (62)$$

(ii) *$\max |\mathcal{K}_t - cI|^2$ is a monotone decreasing function of t ;*

(iii) *If $\mathcal{L}(h_t, k)$ is bounded below, i.e., $\mathcal{L}(h_t, k) \geq A > -\infty$ for some real constant A and $0 \leq t < \infty$, then*

$$\max_X |\mathcal{K}_t - cI|^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (63)$$

Proof: From the proof of Lemma 4.5 we know that

$$\frac{d}{dt}\mathcal{L}(h_t, k) = (\mathcal{K}_t - c h_t, \partial_t h_t). \quad (64)$$

Since h_t is a solution of the evolution equation, we get

$$\frac{d}{dt}\mathcal{L}(h_t, k) = -(\mathcal{K}_t - c h_t, \mathcal{K}_t - c h_t) = -\|\mathcal{K}_t - c h_t\|^2 \quad (65)$$

and (i) follows from the definition of the Riemannian structure in $\text{Herm}^+(\mathfrak{E})$ (considered this time as a metric for endomorphisms). The proofs of (ii) and (iii) are similar to the proof in the classical case [10], but we need to work this time with the operator $\tilde{\square}_h = i\Lambda \mathcal{D}'' \mathcal{D}'_h$ instead of the operator $\square_h = i\Lambda D'' D'_h$. In fact, from this definition $\tilde{\square} v_t = \partial_t \mathcal{K}_t$ and since $v_t = -(\mathcal{K}_t - cI)$, we get

$$(\tilde{\square} + \partial_t) \mathcal{K}_t = 0. \quad (66)$$

On the other hand,

$$\begin{aligned} \mathcal{D}'' \mathcal{D}' |\mathcal{K}_t - cI|^2 &= \mathcal{D}'' \mathcal{D}' \text{tr}(\mathcal{K}_t - cI)^2 \\ &= 2 \text{tr}((\mathcal{K}_t - cI) \mathcal{D}'' \mathcal{D}' \mathcal{K}_t) + 2 \text{tr}(\mathcal{D}'' \mathcal{K}_t \cdot \mathcal{D}' \mathcal{K}_t) \end{aligned}$$

and by taking the trace with respect to ω we get

$$\begin{aligned} \tilde{\square} |\mathcal{K}_t - cI|^2 &= 2 \text{tr}((\mathcal{K}_t - cI) \tilde{\square} \mathcal{K}_t) + 2 i\Lambda \text{tr}(\mathcal{D}'' \mathcal{K}_t \cdot \mathcal{D}' \mathcal{K}_t) \\ &= -2 \text{tr}((\mathcal{K}_t - cI) \partial_t \mathcal{K}_t) - 2 |\mathcal{D}'' \mathcal{K}_t|^2 \\ &= -\partial_t |\mathcal{K}_t - cI|^2 - 2 |\mathcal{D}'' \mathcal{K}_t|^2. \end{aligned}$$

So, finally we obtain

$$(\partial_t + \tilde{\square}) |\mathcal{K}_t - cI|^2 = -2 |\mathcal{D}'' \mathcal{K}_t|^2 \leq 0 \quad (67)$$

and (ii) follows from the maximum principle. Finally, (iii) follows from (ii) and (i) in a similar way to the classical case (see [10] for details). Q.E.D.

At this point we introduce the main result of the section. This establishes a relation among the boundedness property of Donaldson's functional, the semistability and the existence of approximate Hermitian-Yang-Mills structures.

Theorem 5.3 *Let \mathfrak{E} be a Higgs bundle over a compact Kähler manifold X with Kähler form ω . Then we have the implications (i) \rightarrow (ii) \rightarrow (iii) for the following statements:*

- (i) *for any fixed Hermitian structure k in \mathfrak{E} , there exists a constant B such that $\mathcal{L}(h, k) \geq B$ for all Hermitian structures h in \mathfrak{E} ;*
- (ii) *\mathfrak{E} admits an approximate Hermitian-Yang-Mills structure, i.e., given $\epsilon > 0$ there exists an Hermitian structure h in \mathfrak{E} such that*

$$\max_X |\mathcal{K} - ch| < \epsilon; \quad (68)$$

- (iii) *\mathfrak{E} is ω -semistable.*

Proof: Assume (i). Then the functional is bounded below by a constant A . In particular $\mathcal{L}(h_t, k) \geq A$ for h_t , $0 \leq t < \infty$, a one-parameter family of Hermitian structures satisfying the evolution equation. Therefore, from (63) it follows that given $\epsilon > 0$ there exists t_0 such that⁸

$$\max_X |\mathcal{K}_t - cI| < \epsilon \quad \text{for } t > t_0. \quad (69)$$

This shows that (i) implies (ii). On the other hand, (ii) \rightarrow (iii) has been proved by Bruzzo and Graña-Otero in [4]. Q.E.D.

6 Semistable Higgs bundles

We need some results which allow us to solve some problems about Higgs bundles by induction on the rank. This section is essentially a natural extension to Higgs bundles of the classical case, which is explained in detail in [10].

Let

$$0 \longrightarrow \mathfrak{E}' \xrightarrow{\iota} \mathfrak{E} \xrightarrow{p} \mathfrak{E}'' \longrightarrow 0 \quad (70)$$

be an exact sequence of Higgs bundles over a Kähler manifold. As in the ordinary case, an Hermitian structure h in \mathfrak{E} induces Hermitian structures h' and h'' in \mathfrak{E}' and \mathfrak{E}'' respectively. We have also a second fundamental form $A_h \in A^{1,0}(\text{Hom}(E', E''))$ and its adjoint $B_h \in A^{0,1}(\text{Hom}(E'', E'))$ where, as usual, $B_h^* = -A_h$. In a similar way, some properties which hold in the ordinary case, also hold in the Higgs case.

Proposition 6.1 *Given an exact sequence (70) and a pair of Hermitian structures h, k in \mathfrak{E} . Then the function $Q_1(h, k)$ and the form $Q_2(h, k)$ satisfies the following properties:*

- (i) $Q_1(h, k) = Q_1(h', k') + Q_1(h'', k'')$,
- (ii) $Q_2(h, k) = Q_2(h', k') + Q_2(h'', k'') - i \text{tr}[B_h \wedge B_h^* - B_k \wedge B_k^*]$
mod $d' A^{0,1} + d'' A^{1,0}$.

Proof: (i) is straightforward from the definition of Q_1 . On the other hand, (ii) follows from an analysis similar to the ordinary case.

⁸Notice that given $\epsilon > 0$, any metric $h = h_{t_1}$ with $t_1 > t_0$ in principle satisfies (69).

Since the sequence (70) is in particular an exact sequence of holomorphic vector bundles, for any h we have a splitting of the exact sequence by C^∞ -homomorphisms $\mu_h : E \rightarrow E'$ and $\lambda_h : E'' \rightarrow E$. In particular,

$$B_h = \mu_h \circ d'' \circ \lambda_h. \quad (71)$$

We consider now a curve of Hermitian structures $h = h_t$, $0 \leq t \leq 1$ such that $h_0 = k$ and $h_1 = h$. Corresponding to h_t we have a family of homomorphisms μ_t and λ_t . We define the homomorphism $S_t : E'' \rightarrow E'$ given by

$$\lambda_t - \lambda_0 = \iota \circ S_t. \quad (72)$$

A short computation (see [10], Ch.VI) shows that $\partial_t B_t = d''(\partial_t S_t)$ and choosing convenient orthonormal local frames for \mathfrak{E}' and \mathfrak{E}'' , we know that the endomorphism v_t can be represented by the matrix

$$v_t = \begin{pmatrix} v'_t & -\partial_t S_t \\ -(\partial_t S_t)^* & v''_t \end{pmatrix}$$

where v'_t, v''_t are the natural endomorphisms associated to h'_t, h''_t respectively. Now, from the ordinary case we have

$$R_t = \begin{pmatrix} R'_t - B_t \wedge B_t^* & D' B_t \\ -D'' B_t^* & R''_t - B_t^* \wedge B_t \end{pmatrix},$$

where R'_t and R''_t are the Chern curvatures of \mathfrak{E}' and \mathfrak{E}'' associated to the metrics h'_t and h''_t respectively. Now $\mathcal{R}_t^{1,1} = R_t + [\phi, \phi_t]$ and since \mathfrak{E}' and \mathfrak{E}'' are Higgs subbundles of \mathfrak{E} , we obtain a simple expression for the (1,1)-component of the Hitchin-Simpson curvature

$$\mathcal{R}_t^{1,1} = \begin{pmatrix} \mathcal{R}'^{1,1}_t - B_t \wedge B_t^* & D' B_t \\ -D'' B_t^* & \mathcal{R}''^{1,1}_t - B_t^* \wedge B_t \end{pmatrix},$$

where $\mathcal{R}'^{1,1}_t = R'_t + [\phi, \phi_t]_{E'}$ and $\mathcal{R}''^{1,1}_t = R''_t + [\phi, \phi_t]_{E''}$. Hence, at this point we can compute the trace

$$\begin{aligned} \text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) &= \text{tr}(v'_t \cdot \mathcal{R}'^{1,1}_t) + \text{tr}(v''_t \cdot \mathcal{R}''^{1,1}_t) \\ &\quad + \text{tr}(\partial_t S_t \cdot D'' B_t^*) - \text{tr}((\partial_t S_t)^* \cdot D' B_t) \\ &\quad + \text{tr}(v'_t \cdot B_t \wedge B_t^*) - \text{tr}(v''_t \cdot B_t^* \wedge B_t). \end{aligned}$$

The last four terms are exactly the same as in the ordinary case. Finally we get that, modulo an element in $d' A^{0,1} + d'' A^{1,0}$

$$\text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) = \text{tr}(v'_t \cdot \mathcal{R}'^{1,1}_t) + \text{tr}(v''_t \cdot \mathcal{R}''^{1,1}_t) - \partial_t \text{tr}(B_t \wedge B_t^*). \quad (73)$$

Then, multiplying the last expression by i and integrating from $t = 0$ to $t = 1$ we obtain (ii). Q.E.D.

From Proposition 6.1 we get an important result for the compact case when $\mu(\mathfrak{E}) = \mu(\mathfrak{E}') = \mu$. Indeed, in that case we have also $\mu(\mathfrak{E}'') = \mu$. Then, by integrating $Q_1(h, k)$ and $Q_2(h, k)$ over X , and since

$$-i \text{tr}(B \wedge B^*) \wedge \omega^{n-1} = |B|^2 \omega^n / n! \quad (74)$$

we have the same constant c for all functionals $\mathcal{L}(h, k)$, $\mathcal{L}(h', k')$ and $\mathcal{L}(h'', k'')$ and we obtain the following

Corollary 6.2 *Given an exact sequence (70) over a compact Kähler manifold X with $\mu(\mathfrak{E}) = \mu(\mathfrak{E}')$ and a pair of Hermitian structures h and k in \mathfrak{E} , the functional $\mathcal{L}(h, k)$ satisfies the following relation*

$$\mathcal{L}(h, k) = \mathcal{L}(h', k') + \mathcal{L}(h'', k'') + \|B_h\|^2 - \|B_k\|^2. \quad (75)$$

In the one-dimensional case, when X is a compact Riemann surface, the notion of stability (resp. semistability) does not depend on the Kähler form ω , therefore we can establish our results without make reference to any ω . At this point we can establish a boundedness property for the Donaldson functional for semistable Higgs bundles over Riemann surfaces. To be precise we have

Theorem 6.3 *Let \mathfrak{E} be a Higgs bundle over a compact Riemann surface X . If it is semistable, then for any fixed Hermitian structure k in $\text{Herm}^+(\mathfrak{E})$ the set $\{\mathcal{L}(h, k), h \in \text{Herm}^+(\mathfrak{E})\}$ is bounded below.*

Proof: Fix k and assume that \mathfrak{E} is semistable. The proof runs by induction on the rank of \mathfrak{E} . If it is stable, then by [2] there exists an Hermitian-Yang-Mills structure h_0 on it, and we know that Donaldson's functional must attain an absolute minimum at h_0 , i.e., for any other metric h

$$\mathcal{L}(h, k) \geq \mathcal{L}(h_0, k) \quad (76)$$

and hence the set is bounded below. Now, suppose \mathfrak{E} is not stable, then among all proper non-trivial Higgs subsheaves with torsion-free quotient and the same slope as \mathfrak{E} we choose one, say \mathfrak{E}' , with minimal rank. Since $\mu(\mathfrak{E}') = \mu(\mathfrak{E})$ this sheaf is necessarily stable⁹. Now let $\mathfrak{E}'' = \mathfrak{E}/\mathfrak{E}'$, then using Lemma 7.3 in [10] it follows that $\mu(\mathfrak{E}'') = \mu(\mathfrak{E})$ and \mathfrak{E}'' is semistable¹⁰ and hence we have the following exact sequence of sheaves

$$0 \longrightarrow \mathfrak{E}' \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{E}'' \longrightarrow 0 \quad (77)$$

where \mathfrak{E}' and \mathfrak{E}'' are both torsion-free. Since $\dim X = 1$ they are also locally free and hence the sequence is in fact an exact sequence of Higgs bundles. Assume now that h is an arbitrary metric on \mathfrak{E} , then by applying the preceding Corollary to the metrics h and k we obtain

$$\mathcal{L}(h, k) = \mathcal{L}(h', k') + \mathcal{L}(h'', k'') + \|B_h\|^2 - \|B_k\|^2, \quad (78)$$

where h', k' and h'', k'' are the Hermitian structures induced by h, k in \mathfrak{E}' and \mathfrak{E}'' respectively. If the rank of \mathfrak{E} is one, it is stable and hence $\mathcal{L}(h, k)$ is bounded below by a constant which depends on k . Now, by the inductive hypothesis,

⁹If \mathfrak{E}' is not stable, there exists a proper Higgs subsheaf \mathfrak{F}' of \mathfrak{E}' with $\mu(\mathfrak{F}') \geq \mu(\mathfrak{E}')$ and since \mathfrak{F}' is clearly a subsheaf of \mathfrak{E} and this is semistable we necessarily obtain $\mu(\mathfrak{F}') = \mu(\mathfrak{E})$, which is a contradiction, because \mathfrak{E}' was chosen with minimal rank.

¹⁰In fact, if \mathfrak{E}'' is not semistable, then there exists a proper Higgs subsheaf \mathfrak{H} of \mathfrak{E}'' with $\mu(\mathfrak{H}) > \mu(\mathfrak{E}'')$. Then, using Lemma 7.3 in [10] we have $\mu(\mathfrak{E}'') > \mu(\mathfrak{E}''/\mathfrak{H})$. Defining \mathfrak{K} as the kernel of the morphism $\mathfrak{E} \longrightarrow \mathfrak{E}''/\mathfrak{H}$, we get the exact sequence

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{E}''/\mathfrak{H} \longrightarrow 0$$

and since $\mu(\mathfrak{E}) = \mu(\mathfrak{E}'')$, using again the same Lemma in [10] we conclude that $\mu(\mathfrak{K}) > \mu(\mathfrak{E})$, which contradicts the semistability of \mathfrak{E} .

$\mathcal{L}(h', k')$ and $\mathcal{L}(h'', k'')$ are bounded below by constants depending only on k' and k'' respectively. Then $\mathcal{L}(h, k)$ is bounded below by a constant depending on k . Q.E.D.

As a consequence of the above result, we get that in the one-dimensional case all three conditions in the main theorem are equivalent. As a consequence we obtain the following

Corollary 6.4 *Let \mathfrak{E} be a Higgs bundle over a compact Riemann surface X . Then \mathfrak{E} is semistable if and only if \mathfrak{E} admits an approximate Hermitian-Yang-Mills structure.*

This equivalence between the notions of approximate Hermitian-Yang-Mills structures and semistability is one version of the so called *Hitchin-Kobayashi correspondence* for Higgs bundles. As a consequence of the Corollary 6.4 we see that in the one-dimensional case, all results about Higgs bundles written in terms of approximate Hermitian-Yang-Mills structures can be translated in terms of semistability. In particular we have

Corollary 6.5 *If \mathfrak{E}_1 and \mathfrak{E}_2 are semistable Higgs bundles over a compact Riemann surface X , then so is the tensor product $\mathfrak{E}_1 \otimes \mathfrak{E}_2$. Furthermore if $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, so is the Whitney sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2$.*

Corollary 6.6 *If \mathfrak{E} is semistable Higgs bundle over a Riemann surface X , then so is the tensor product bundle $\mathfrak{E}^{\otimes p} \otimes \mathfrak{E}^{*\otimes q}$ and the exterior product bundle $\bigwedge^p \mathfrak{E}$ whenever $p \leq r$.*

The equivalence between the existence of approximate Hermitian-Yang-Mills structures and semistability is also true in higher dimensions. However, since torsion-free sheaves over compact Kähler manifolds with $\dim X \geq 2$ may not be locally free (they are locally free only outside its singularity set) it is necessary to use Higgs sheaves and not only Higgs bundles.

7 Admissible metrics for Higgs sheaves

A natural notion of a metric on a torsion-free sheaf is that of admissible Hermitian structure. This was first introduced by S. Bando and Y.-T. Siu in [19]. In their article, they proved first the existence of admissible structures on any torsion-free sheaf, and then obtained an equivalence between the stability of a torsion-free sheaf and the existence of an admissible Hermitian-Yang-Mills metric on it, thus extending the Hitchin-Kobayashi correspondence to torsion-free sheaves.

Admissible structures were used again by I. Biswas and G. Schumacher [5] to prove an extended version of the correspondence of Bando and Siu to the Higgs case. In this last section, we briefly discuss some of these notions.

Let \mathfrak{E} be a torsion-free Higgs sheaf over a compact Kähler manifold X . The singularity set of \mathfrak{E} is the subset $S = S(\mathfrak{E}) \subset X$ where \mathfrak{E} is not locally free. As is well known, S is a complex analytic subset with $\text{codim} S \geq 2$. Following [5], [19], an *admissible structure* h on \mathfrak{E} is an Hermitian metric on the bundle $\mathfrak{E}|_{X \setminus S}$

with the following two properties:

- (i) The Chern curvature R of h is square-integrable, and
- (ii) The corresponding mean curvature $K = i\Lambda R$ is L^1 -bounded.

Let consider now the natural embedding of \mathfrak{E} into its double dual $\mathfrak{E}^{\vee\vee}$; since $S(\mathfrak{E}^{\vee\vee}) \subset S(\mathfrak{E})$, an admissible structure on $\mathfrak{E}^{\vee\vee}$ restricts to an admissible structure on \mathfrak{E} . An admissible structure h is called an *admissible Hermitian-Yang-Mills structure* if on $X \setminus S$ the mean curvature of the Hitchin-Simpson connection is proportional to the identity. In other words if

$$\mathcal{K} = K + i\Lambda[\phi, \bar{\phi}_h] = c \cdot I \quad (79)$$

is satisfied on $X \setminus S$ for some constant c , where I is the identity endomorphism of E . It is important to note here that, in contrast to an admissible metric, the condition defining an admissible Hermitian-Yang-Mills metric depends on the Higgs field.

Let \mathfrak{E} be a torsion-free Higgs sheaf over a compact Kähler manifold X . Since its singularity set S is a complex analytic subset with codimension greater or equal than two, $X \setminus S$ satisfies all assumptions Simpson [2] imposes¹¹ on the base manifold and hence we can see $\mathfrak{E}|_{X \setminus S}$ as a Higgs bundle over the non-compact Kähler manifold $X \setminus S$.

Following Simpson [2], Proposition 3.3 (see also [5], Corollary 3.5) it follows that a torsion-free Higgs sheaf over a compact Kähler manifold with an admissible Hermitian-Yang-Mills metric must be at least semistable. However, as Biswas and Schumacher have shown in [5], this is just a part of a stronger result. To be more precise, they proved the Hitchin-Kobayashi correspondence for Higgs sheaves. This result can be written as

Theorem 7.1 *Let \mathfrak{E} be a torsion-free Higgs sheaf over a compact Kähler manifold X with Kähler form ω . Then, it is ω -polystable if and only if there exists an admissible Hermitian-Yang-Mills structure on it.*

Let h be an admissible metric on \mathfrak{E} , then K_h is L^1 -bounded. On the other hand, from [5], Lemma 2.6, we know the Higgs field ϕ is also L^1 -bounded on $X \setminus S$ (in particular it is square integrable). From this we conclude that

$$\mathcal{K}_h = K_h + i\Lambda[\phi, \bar{\phi}_h] \quad (80)$$

is L^1 -bounded and hence, for any admissible metric h on the torsion-free Higgs sheaf \mathfrak{E} we must have

$$\int_{X \setminus S} |\mathcal{K}_h| \omega^n < \infty. \quad (81)$$

Let $\text{Herm}^+(\mathfrak{E}_{X \setminus S})$ be the space of all smooth metrics on $\mathfrak{E}_{X \setminus S}$ satisfying the condition (81) and suppose that h and k are two metrics in the same connected

¹¹Notice that since X is compact, by [2], Proposition 2.1, it satisfies the assumptions on the base manifold that Simpson introduced. Now, from this and [2], Proposition 2.2, it follows that $X \setminus S$ also satisfies the assumptions.

component of $\text{Herm}^+(\mathfrak{E}_{X \setminus S})$. Then $h = ke^v$ for some endomorphism v of $E|_{X \setminus S}$ and following Simpson [2], we can write the Donaldson functional as

$$\mathcal{L}(ke^v, k) = \int_{X \setminus S} \text{tr}[v(\mathcal{K}_k - cI)] \frac{\omega^n}{n!} + \int_{X \setminus S} \sum_{i,j=1}^r \psi(\lambda_i, \lambda_j) |\mathcal{D}'' v_j^i|^2 \frac{\omega^{n-1}}{(n-1)!} \quad (82)$$

where the function ψ is given by

$$\psi(\lambda_i, \lambda_j) = \frac{e^{(\lambda_i - \lambda_j)} - (\lambda_i - \lambda_j) - 1}{(\lambda_i - \lambda_j)^2}. \quad (83)$$

We define the *Donaldson functional on the Higgs sheaf* \mathfrak{E} just as the corresponding functional (82) defined on the Higgs bundle $\mathfrak{E}|_{X \setminus S}$. In [2], Simpson established an inequality between the supremum of the norm of the endomorphism v relating the metrics h and k and the Donaldson functional for Higgs bundles over (non necessarily) compact Kähler manifolds; this result can be immediately adapted to Higgs sheaves as follows:

Corollary 7.2 *Let k be an admissible metric on a torsion-free Higgs sheaf \mathfrak{E} over a compact Kähler manifold X with Kähler form ω and suppose that $\sup_{X \setminus S} |\mathcal{K}_k| \leq B$ for certain fixed constant B . If \mathfrak{E} is ω -stable, then there exist constants C_1 and C_2 such that*

$$\sup_{X \setminus S} |v| \leq C_1 + C_2 \mathcal{L}(ke^v, k) \quad (84)$$

for any selfadjoint endomorphism v with $\text{tr } v = 0$ and $\sup_{X \setminus S} |v| < \infty$ and such that $\sup_{X \setminus S} |\mathcal{K}_{ke^v}| \leq B$.

In a future work, we will study more in detail admissible metrics and Donaldson's functional for torsion-free Higgs sheaves and the equivalence between semistability and the existence of approximate Hermitian-Yang-Mills metrics for Higgs bundles in higher dimensions.

References

- [1] N. J. HITCHIN, *The self-duality equations on a Riemann surface*, Proc. London. Math., **55** (1987), 59-126.
- [2] C. T. SIMPSON, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988) 867-918.
- [3] C. T. SIMPSON, *Higgs bundles and local systems*, Publ. Math. I.H.E.S., **75** (1992), 5-92.
- [4] U. BRUZZO AND B. GRAÑA OTERO, *Metrics on semistable and numerically effective Higgs bundles*, J. reine ang. Math., **612** (2007), 59-79.
- [5] I. BISWAS AND G. SCHUMACHER, *Yang-Mills equations for stable Higgs sheaves*, International Journal of Mathematics, Vol. 20, No 5 (2009), 541-556.

- [6] S. B. BRADLOW AND T. GOMEZ, *Extensions of Higgs bundles*, Illinois Journal of Mathematics, **46** (2002), 587-625.
- [7] S. B. BRADLOW, O. GARCIA-PRADA AND I. MUNDET I RIERA, *Relative Hitchin-Kobayashi correspondence for principal pairs*, Quarterly Journal of Mathematics, **54** (2003), 171-208.
- [8] S. KOBAYASHI, *First Chern class and holomorphic tensor fields*, Nagoya Math. J **77** (1980), 5-11.
- [9] S. KOBAYASHI, *Curvature and stability of vector bundles*, Proc. Jap. Acad. **58** (1982), 158-162.
- [10] S. KOBAYASHI, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, Vol. 15 (Iwanami Shoten Publishers and Princeton University Press, 1987).
- [11] K. UHLENBECK AND S.-T. YAU, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Commun. Pure Appl. Math., **39** (1986), pp. S257-S293.
- [12] M. NARASIMHAN AND C. SESHADRI, *Stable and unitary bundles on a compact Riemann surface*, Math. Ann., **82** (1965), 540-564.
- [13] Y.-T. SIU, *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, Birkhäuser, Basel-Boston, (1987).
- [14] M. LÜBKE, *The Kobayashi-Hitchin correspondence*, World Scientific Publishing Co. Pte. Ltd., (1995).
- [15] M. LÜBKE, *Stability of Einstein-Hermitian vector bundles*, Manuscripta Math., **42** (1983), pp. 245-257.
- [16] S. K. DONALDSON, *A new proof of a theorem of Narasimhan and Seshadri*, J. Diff. Geom. Soc., **18** (1983), pp. 269-278.
- [17] S. K. DONALDSON, *Anti-self-dual Yang-Mills connections on complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc., **3** (1985), pp. 1-26.
- [18] S. K. DONALDSON, *Infinite determinants, stable bundles and curvature*, Duke Math. J **54** (1987), pp. 231-247.
- [19] S. BANDO AND Y.-T. SIU, *Stable sheaves and Einstein-Hermitian metrics*, Geometry and analysis on complex manifolds, (World Scientific Publishing, River Edge, NJ, 1994), pp. 39-50.
- [20] N. M. BUCHDAHL, *Hermitian-Einstein connections and stable vector bundles over compact complex surfaces*, Math. Ann. **280** (1988), pp. 625-648. pp. 503-547.
- [21] R. LENA, *Positività di Fibrati di Higgs*, M.Sc. Thesis, Università di Trieste and SISSA, (2010).
- [22] A. JACOB, *Existence of approximate Hermitian-Einstein structures on semistable bundles*, arXiv:1012.1888v1, 2010.